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On the Hausdorff measure of noncompactness for the parameterized Prokhorov metric

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Abstract
We quantify the Prokhorov theorem by establishing an explicit formula for the Hausdorff measure of noncompactness (HMNC) for the parameterized Prokhorov metric on the set of Borel probability measures on a Polish space. Furthermore, we quantify the Arzelà-Ascoli theorem by obtaining upper and lower estimates for the HMNC for the uniform norm on the space of continuous maps of a compact interval into Euclidean N-space, using Jung's theorem on the Chebyshev radius. Finally, we combine the obtained results to quantify the stochastic Arzelà-Ascoli theorem by providing upper and lower estimates for the HMNC for the parameterized Prokhorov metric on the set of multivariate continuous stochastic processes.

Keywords: Arzelà-Ascoli theorem; Chebyshev radius; Hausdorff measure of noncompactness; Jung's theorem; parametrized Prokhorov metric; Prokhorov's theorem; stochastic Arzelà-Ascoli theorem

1 Introduction and statement of the main results

For the basic probabilistic concepts and results, we refer the reader to any standard work on probability theory such as [1].

Let S be a Polish space, that is, a separable completely metrizable topological space, and $\mathcal{P}(S)$ the collection of Borel probability measures on S, equipped with the weak topology τ_w, that is, the weakest topology for which each map

$$\mathcal{P}(S) \to \mathbb{R} : P \mapsto \int f\, dP$$

with bounded and continuous $f : S \to \mathbb{R}$ is continuous. The space $\mathcal{P}(S)$ is known to be Polish.

We call a collection $\Gamma \subset \mathcal{P}(S)$ uniformly tight iff for each $\epsilon > 0$, there exists a compact set $K \subset S$ such that $P(S \setminus K) < \epsilon$ for all $P \in \Gamma$.

The following celebrated result interrelates the τ_w-relative compactness with uniform tightness.

Theorem 1.1 (Prokhorov) *A collection $\Gamma \subset \mathcal{P}(S)$ is τ_w-relatively compact if and only if it is uniformly tight.*

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Fix $N \in \mathbb{N}_0$ and let $\mathcal{C}$ be the space of continuous maps x of the compact interval $[0,1]$ into Euclidean N-space $\mathbb{R}^N$ equipped with the uniform topology τ_∞, that is, the topology derived from the uniform norm

$$\|x\|_\infty = \sup_{t\in[0,1]} \bigl|x(t)\bigr|,$$

where $|\cdot|$ stands for the Euclidean norm. The space $\mathcal{C}$ is also known to be Polish.

Recall that a set $\mathcal{X} \subset \mathcal{C}$ is said to be uniformly bounded iff there exists a constant $M > 0$ such that $|x(t)| \leq M$ for all $x \in \mathcal{X}$ and $t \in [0,1]$, and uniformly equicontinuous iff for each $\epsilon > 0$, there exists $\delta > 0$ such that $|x(s) - x(t)| < \epsilon$ for all $x \in \mathcal{X}$ and $s,t \in [0,1]$ with $|s-t| < \delta$.

In this setting, the following theorem is a classical result [2].

Theorem 1.2 (Arzelà-Ascoli) *A collection $\mathcal{X} \subset \mathcal{C}$ is τ_∞-relatively compact if and only if it is uniformly bounded and uniformly equicontinuous.*

Let $\Omega = (\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. Throughout, a continuous stochastic process (c.s.p.) is a Borel-measurable map of Ω into $\mathcal{C}$, and we consider on the set of c.s.p.s the weak topology τ_w, that is, the topology with open sets $\{\xi \text{ c.s.p.} \mid \mathbb{P}_\xi \in \mathcal{G}\}$, where $\mathbb{P}_\xi$ is the probability distribution of ξ, and $\mathcal{G}$ is a τ_w-open set in $\mathcal{P}(\mathcal{C})$.

A collection Ξ of c.s.p.s is said to be stochastically uniformly bounded iff for each $\epsilon > 0$, there exists $M > 0$ such that $\mathbb{P}(\|\xi\|_\infty > M) < \epsilon$ for all $\xi \in \Xi$, and stochastically uniformly equicontinuous iff for all $\epsilon, \epsilon' > 0$, there exists $\delta > 0$ such that $\mathbb{P}(\sup_{|s-t|<\delta} |\xi(s) - \xi(t)| \geq \epsilon) < \epsilon'$ for all $\xi \in \Xi$, the supremum taken over all $s,t \in [0,1]$ for which $|s-t| < \delta$.

It is not hard to see that combining Theorem 1.1 and Theorem 1.2 yields the following stochastic version of Theorem 1.2, which plays a crucial role in the development of functional central limit theory.

Theorem 1.3 (Stochastic Arzelà-Ascoli) *A collection Ξ of c.s.p.s is τ_w-relatively compact if and only if it is stochastically uniformly bounded and stochastically uniformly equicontinuous.*

In a complete metric space (X,d), the Hausdorff measure of noncompactness of a set $A \subset X$ [3, 4] is given by

$$\mu_{\mathrm{H},d}(A) = \inf_F \sup_{x\in A} \inf_{y\in F} d(x,y),$$

the first infimum running through all finite sets $F \subset X$. It is well known that A is d-bounded if and only if $\mu_{\mathrm{H},d}(A) < \infty$, and d-relatively compact if and only if $\mu_{\mathrm{H},d}(A) = 0$.

Fix a complete metric d metrizing the topology of the Polish space S. The Prokhorov distance with parameter $\lambda \in \mathbb{R}_0^+$ between probability measures $P, Q \in \mathcal{P}(S)$ [5] is defined as the infimum of all numbers $\alpha \in \mathbb{R}_0^+$ such that

$$P(A) \leq Q\bigl(A^{(\lambda\alpha)}\bigr) + \alpha$$

for all Borel sets $A \subset S$, where

$$A^{(\epsilon)} = \Bigl\{x \in S \Bigm| \inf_{a\in A} d(a,x) \leq \epsilon\Bigr\}.$$

This distance is denoted by $\rho_\lambda(P,Q)$. It defines a complete metric on $\mathcal{P}(S)$ and induces the weak topology τ_w. It is also known that $\rho_{\lambda_1} \leq \rho_{\lambda_2}$ if $\lambda_1 \geq \lambda_2$ and that

$$\sup_{\lambda\in\mathbb{R}_0^+} \rho_\lambda(P,Q) = \sup_A \bigl|P(A) - Q(A)\bigr|,$$

the supremum being taken over all Borel sets $A \subset S$.

For a collection $\Gamma \subset \mathcal{P}(S)$, we define the measure of nonuniform tightness as

$$\mu_{\mathrm{ut}}(\Gamma) = \sup_{\epsilon>0} \inf_Y \sup_{P\in\Gamma} P\biggl(S \setminus \bigcup_{y\in Y} B(y,\epsilon)\biggr),$$

where the infimum runs through all finite sets $Y \subset S$, and

$$B(y,\epsilon) = \bigl\{x \in S \mid d(y,x) < \epsilon\bigr\}.$$

It is clear that $\mu_{\mathrm{ut}}(\Gamma) = 0$ if Γ is uniformly tight. The converse holds as well. Indeed, suppose that $\mu_{\mathrm{ut}}(\Gamma) = 0$ and fix $\epsilon > 0$. Then, for each $n \in \mathbb{N}_0$, choose a finite set $Y_n \subset S$ such that

$$P\biggl(S \setminus \bigcup_{y\in Y_n} B(y,1/n)\biggr) < \epsilon/2^n$$

for all $P \in \Gamma$. Put

$$K = \bigcap_{n\in\mathbb{N}_0} \bigcup_{y\in Y_n} B^\star(y,1/n)$$

with $B^\star(y,1/n)$ the closure of $B(y,1/n)$. Then K is a compact set such that $P(S \setminus K) < \epsilon$ for all $P \in \Gamma$. We conclude that Γ is uniformly tight. The measure μ_{ut} is slightly weaker than the weak measure of tightness studied in [6].

By the previous considerations we know that a set $\Gamma \subset \mathcal{P}(S)$ is τ_w-relatively compact if and only if $\mu_{\mathrm{H},\rho_\lambda}(\Gamma) = 0$ for each $\lambda \in \mathbb{R}_0^+$, and uniformly tight if and only if $\mu_{\mathrm{ut}}(\Gamma) = 0$. Therefore, Theorem 1.4, our first main result, which provides a quantitative relation between the numbers $\mu_{\mathrm{H},\rho_\lambda}(\Gamma)$ and $\mu_{\mathrm{ut}}(\Gamma)$, is a strict generalization of Theorem 1.1. The proof is given in Section 2.

Theorem 1.4 (Quantitative Prokhorov) *For a collection* $\Gamma \subset \mathcal{P}(S)$,

$$\sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\rho_\lambda}(\Gamma) = \mu_{\mathrm{ut}}(\Gamma).$$

From now on, we consider on the space $\mathcal{C}$ the uniform metric derived from the uniform norm, and for a set $\mathcal{X} \subset \mathcal{C}$, we let $\mu_{\mathrm{H},\infty}(\mathcal{X})$ stand for the Hausdorff measure of noncompactness; more precisely,

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) = \inf_{\mathcal{F}} \sup_{x\in\mathcal{X}} \inf_{y\in\mathcal{F}} \|x-y\|_\infty,$$

the infimum taken over all finite sets $\mathcal{F} \subset \mathcal{C}$. Clearly, $\mathcal{X}$ is τ_∞-relatively compact if and only if $\mu_{\mathrm{H},\infty}(\mathcal{X}) = 0$.

The measure of nonuniform equicontinuity of $\mathcal{X} \subset \mathcal{C}$ is defined by

$$\mu_{\mathrm{uec}}(\mathcal{X}) = \inf_{\delta>0} \sup_{x\in\mathcal{X}} \sup_{|s-t|<\delta} \left|x(s) - x(t)\right|,$$

the second supremum running through all $s, t \in [0,1]$ with $|s-t| < \delta$. We readily see that $\mathcal{X}$ is uniformly equicontinuous if and only if $\mu_{\mathrm{uec}}(\mathcal{X}) = 0$. In [3], it was shown that μ_{uec} is a measure of noncompactness on the space $\mathcal{C}$ (Theorem 11.2).

Theorem 1.5, our second main result, entails that the measures $\mu_{\mathrm{H},\infty}$ and μ_{uec} are Lipschitz equivalent on the collection of uniformly bounded subsets of $\mathcal{C}$, and thus it strictly generalizes Theorem 1.2. The proof, which hinges upon a classical result of Jung on the Chebyshev radius, is given in Section 3.

Theorem 1.5 (Quantitative Arzelà-Ascoli) *For* $\mathcal{X} \subset \mathcal{C}$,

$$\frac{1}{2}\mu_{\mathrm{uec}}(\mathcal{X}) \leq \mu_{\mathrm{H},\infty}(\mathcal{X}).$$

Suppose, in addition, that $\mathcal{X}$ *is uniformly bounded. Then*

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X}).$$

In particular, if $N = 1$, *then*

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) = \frac{1}{2}\mu_{\mathrm{uec}}(\mathcal{X}),$$

and, regardless of N,

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \frac{\sqrt{2}}{2}\mu_{\mathrm{uec}}(\mathcal{X}).$$

We transport the parameterized Prokhorov metric from $\mathcal{P}(\mathcal{C})$ to the collection of c.s.p.s via their probability distributions. Thus, for c.s.p.s ξ and η,

$$\rho_\lambda(\xi,\eta) = \rho_\lambda(\mathbb{P}_\xi, \mathbb{P}_\eta).$$

Note that a set of c.s.p.s Ξ is τ_ω-relatively compact if and only if $\mu_{\mathrm{H},\rho_\lambda}(\Xi) = 0$ for all $\lambda \in \mathbb{R}_0^+$.

For a set of c.s.p.s Ξ, the measure of nonstochastic uniform boundedness is given by

$$\mu_{\mathrm{sub}}(\Xi) = \inf_{M\in\mathbb{R}_0^+} \sup_{\xi\in\Xi} \mathbb{P}\bigl(\|\xi\|_\infty > M\bigr),$$

and the measure of nonstochastic uniform equicontinuity by

$$\mu_{\mathrm{suec}}(\Xi) = \sup_{\epsilon>0} \inf_{\delta>0} \sup_{\xi\in\Xi} \mathbb{P}\Bigl(\sup_{|s-t|<\delta} \left|\xi(s) - \xi(t)\right| \geq \epsilon\Bigr),$$

where the third supremum is taken over all $s, t \in [0,1]$ with $|s-t| < \delta$. It is easily seen that Ξ is stochastically uniformly bounded if and only if $\mu_{\mathrm{sub}}(\Xi) = 0$, and stochastically uniformly equicontinuous if and only if $\mu_{\mathrm{suec}}(\Xi) = 0$. The measure μ_{suec} was studied in [6].

In Section 4, we prove that combining Theorem 1.4 and Theorem 1.5 leads to Theorem 1.6, our third main result, which gives upper and lower bounds for $\sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\rho_\lambda}$ in terms of μ_{sub} and μ_{suec}. Theorem 1.6 strictly generalizes Theorem 1.3.

Theorem 1.6 (Quantitative stochastic Arzelà-Ascoli) *Let Ξ be a collection of c.s.p.s. Then*

$$\max\bigl\{\mu_{\mathrm{sub}}(\Xi), \mu_{\mathrm{suec}}(\Xi)\bigr\} \le \sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\rho_\lambda}(\Xi) \le \mu_{\mathrm{sub}}(\Xi) + \mu_{\mathrm{suec}}(\Xi).$$

In particular, if Ξ is stochastically uniformly bounded, then

$$\sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\rho_\lambda}(\Xi) = \mu_{\mathrm{suec}}(\Xi),$$

and, if Ξ is stochastically uniformly equicontinuous, then

$$\sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\rho_\lambda}(\Xi) = \mu_{\mathrm{sub}}(\Xi).$$

2 Proof of Theorem 1.4

For a collection $\Gamma \subset \mathcal{P}(S)$, put

$$p_\Gamma = \sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\rho_\lambda}(\Gamma)$$

and

$$t_\Gamma = \mu_{\mathrm{ut}}(\Gamma).$$

We first show that

$$p_\Gamma \le t_\Gamma$$

with an argument that essentially refines the first part of the proof of Theorem 4.9 in [6].

Fix $\lambda \in \mathbb{R}_0^+$, $\epsilon > 0$, and choose pairwise disjoint Borel sets

$$A_1, \dots, A_n \subset S$$

with diameters less than $\lambda\epsilon$ such that

$$\forall P \in \Gamma: \quad P\left(S \setminus \bigcup_{i=1}^n A_i\right) \le t_\Gamma + \epsilon/2.$$

Then, for each $i \in \{1, \dots, n\}$, pick $x_i \in A_i$, and, assuming without loss of generality that $S \setminus \bigcup_{i=1}^n A_i$ is nonempty, $x_{n+1} \in S \setminus \bigcup_{i=1}^n A_i$. Finally, fix $m \in \mathbb{N}_0$ such that

$$n/m \le \epsilon/2,$$

and let Φ be a finite collection of Borel probability measures on S of the form

$$Q = \sum_{i=1}^{n+1} (k_i/m)\delta_{x_i},$$

where the k_i range in $\{0, \dots, m\}$ so that

$$\sum_{i=1}^{n+1} k_i = m,$$

and δ_{x_i} stands for the Dirac probability measure putting all its mass on x_i.

We now claim that

$$\forall P \in \Gamma, \exists Q \in \Phi: \quad \rho_\lambda(P, Q) \leq t_\Gamma + \epsilon,$$

which finishes the proof of the desired inequality.

To prove the claim, take $P \in \Gamma$ and construct

$$Q = \sum_{i=1}^{n+1} (k_i/m)\delta_{x_i}$$

in Φ such that

$$P(A_i) \leq k_i/m + 1/m$$

for all $i \in \{1, \dots, n\}$. For a Borel set $A \subset S$, let I stand for the set of those $i \in \{1, \dots, n\}$ for which $A_i \cap A$ is nonempty. Then we derive from the calculation

$$\begin{aligned} P(A) &\leq P\left(\bigcup_{i \in I} A_i\right) + P\left(S \setminus \bigcup_{i=1}^{n} A_i\right) \\ &\leq \sum_{i \in I} P(A_i) + t_\Gamma + \epsilon/2 \\ &\leq \sum_{i \in I} (k_i/m + 1/m) + t_\Gamma + \epsilon/2 \\ &\leq Q\left(\bigcup_{i \in I} A_i\right) + n/m + t_\Gamma + \epsilon/2 \\ &\leq Q\bigl(A^{(\lambda(t_\Gamma + \epsilon))}\bigr) + t_\Gamma + \epsilon \end{aligned}$$

that

$$\rho_\lambda(P, Q) \leq t_\Gamma + \epsilon,$$

establishing the claim.

We now show that

$$t_\Gamma \leq p_\Gamma.$$

Fix $\epsilon, \epsilon' > 0$. Choose $\lambda \in \mathbb{R}_0^+$ such that

$$\lambda\bigl(p_\Gamma + \epsilon'/2\bigr) \le \epsilon/2$$

and take a finite collection $\Phi \subset \mathcal{P}(S)$ such that, for each $P \in \Gamma$, there exists $Q \in \Phi$ for which

$$\rho_\lambda(P, Q) \le \mu_{\mathrm{H},\rho_\lambda}(\Gamma) + \epsilon'/2 \le p_\Gamma + \epsilon'/2.$$

The collection Φ being finite, we can pick a finite set $Y \subset S$ such that

$$\forall Q \in \Phi: \quad Q\biggl(S \setminus \bigcup_{y \in Y} B(y, \epsilon/2)\biggr) \le \epsilon'/2.$$

We claim that

$$\forall P \in \Gamma: \quad P\biggl(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\biggr) \le p_\Gamma + \epsilon',$$

proving the desired inequality.

To establish the claim, take $P \in \Gamma$, and let Q be a probability measure in Φ such that

$$\rho_\lambda(P, Q) \le p_\Gamma + \epsilon'/2.$$

Then

$$\begin{aligned}
&P\biggl(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\biggr) \\
&\quad \le Q\biggl(\biggl(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\biggr)^{(\lambda(p_\Gamma + \epsilon'/2))}\biggr) + p_\Gamma + \epsilon'/2 \\
&\quad \le Q\biggl(\biggl(S \setminus \bigcup_{y \in Y} B(y, \epsilon)\biggr)^{(\epsilon/2)}\biggr) + p_\Gamma + \epsilon'/2 \\
&\quad \le Q\biggl(S \setminus \bigcup_{y \in Y} B(y, \epsilon/2)\biggr) + p_\Gamma + \epsilon'/2 \\
&\quad \le p_\Gamma + \epsilon',
\end{aligned}$$

which finishes the proof of the claim.

3 Proof of Theorem 1.5

Before writing down the proof of Theorem 1.5, we give the required preparation.

For a bounded set $A \subset \mathbb{R}^N$, its diameter is given by

$$\operatorname{diam}(A) = \sup_{x,y \in A} |x - y|,$$

and the Chebyshev radius by

$$r(A) = \inf_{x\in\mathbb{R}^N} \sup_{y\in A} |x-y|.$$

It is well known that, for each bounded set $A \subset \mathbb{R}^N$, there exists a unique $x_A \in \mathbb{R}^N$ such that

$$\sup_{y\in A} |x_A - y| = r(A).$$

The point x_A is called the Chebyshev center of A. A good exposition of the previous notions in a general normed vector space can be found in [7], Section 33.

Theorem 3.1 provides a relation between the diameter and the Chebyshev radius of a bounded set in $\mathbb{R}^N$. A beautiful proof can be found in [8]. For extensions of the result, we refer to [9–11], and [12].

Theorem 3.1 (Jung) *Let $A \subset \mathbb{R}^N$ be a bounded set. Then*

$$\frac{1}{2}\operatorname{diam}(A) \leq r(A) \leq \left(\frac{N}{2N+2}\right)^{1/2} \operatorname{diam}(A).$$

We need two additional simple lemmas on linear interpolation.

For $c_0 \in \mathbb{R}^N$ and $r \in \mathbb{R}_0^+$, we denote by $B^\star(c_0, r)$ the closed ball with center c_0 and radius r.

Lemma 3.2 *Consider $c_1, c_2 \in \mathbb{R}^N$ and $r \in \mathbb{R}_0^+$, and assume that*

$$B^\star(c_1, r) \cap B^\star(c_2, r) \neq \emptyset.$$

Let L be the $\mathbb{R}^N$-valued map on the compact interval $[\alpha, \beta]$ defined by

$$L(t) = \frac{\beta - t}{\beta - \alpha} c_1 + \frac{t-\alpha}{\beta-\alpha} c_2.$$

Then, for all $t \in [\alpha, \beta]$ and $y \in B^\star(c_1, r) \cap B^\star(c_2, r)$,

$$\bigl|L(t) - y\bigr| \leq r.$$

Proof The calculation

$$\begin{aligned}
\bigl|L(t) - y\bigr| &= \left|\frac{\beta - t}{\beta - \alpha}(c_1 - y) + \frac{t-\alpha}{\beta-\alpha}(c_2 - y)\right| \\
&\leq \frac{\beta - t}{\beta - \alpha}|c_1 - y| + \frac{t-\alpha}{\beta-\alpha}|c_2 - y| \\
&\leq \frac{\beta - t}{\beta - \alpha} r + \frac{t-\alpha}{\beta-\alpha} r \\
&= r
\end{aligned}$$

proves the lemma. □

Lemma 3.3 *Consider $c_1, c_2, y_1, y_2 \in \mathbb{R}^N$ and $\epsilon > 0$, and suppose that*

$$|c_1 - y_1| \leq \epsilon$$

and

$$|c_2 - y_2| \leq \epsilon.$$

Let L and M be the $\mathbb{R}^N$-valued maps on the compact interval $[\alpha, \beta]$ defined by

$$L(t) = \frac{\beta - t}{\beta - \alpha} c_1 + \frac{t - \alpha}{\beta - \alpha} c_2$$

and

$$M(t) = \frac{\beta - t}{\beta - \alpha} y_1 + \frac{t - \alpha}{\beta - \alpha} y_2.$$

Then

$$\|L - M\|_\infty \leq \epsilon.$$

Proof It is analogous to the proof of Lemma 3.2. □

Proof of Theorem 1.5 We first prove that

$$\frac{1}{2}\mu_{\text{uec}}(\mathcal{X}) \leq \mu_{\text{H},\infty}(\mathcal{X}).$$

Let $\alpha > 0$ be such that $\mu_{\text{H},\infty}(\mathcal{X}) < \alpha$. Then there exists a finite set $\mathcal{F} \subset \mathcal{C}$ such that, for all $x \in \mathcal{X}$, there exists $y \in \mathcal{F}$ for which $\|y - x\|_\infty \leq \alpha$. Take $\epsilon > 0$. Since $\mathcal{F}$ is uniformly equicontinuous, there exists $\delta > 0$ such that

$$\forall y \in \mathcal{F}, \forall s, t \in [0,1]: \quad |s - t| < \delta \quad \Rightarrow \quad \bigl|y(s) - y(t)\bigr| \leq \epsilon. \tag{1}$$

Now, for $x \in \mathcal{X}$, choose $y \in \mathcal{F}$ such that

$$\|y - x\|_\infty \leq \alpha. \tag{2}$$

Then, for $s, t \in [0,1]$ with $|s - t| < \delta$, we have, by (1) and (2),

$$\bigl|x(s) - x(t)\bigr| \leq \bigl|x(s) - y(s)\bigr| + \bigl|y(s) - y(t)\bigr| + \bigl|y(t) - x(t)\bigr| \leq 2\alpha + \epsilon,$$

which, by the arbitrariness of ϵ, reveals that $\mu_{\text{uec}}(\mathcal{X}) \leq 2\alpha$, and thus, by the arbitrariness of α, we have the inequality

$$\frac{1}{2}\mu_{\text{uec}}(\mathcal{X}) \leq \mu_{\text{H},\infty}(\mathcal{X}).$$

Next, assume that $\mathcal{X} \subset \mathcal{C}$ is uniformly bounded. We show that

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X}).$$

Fix $\epsilon > 0$. Then, $\mathcal{X}$ being uniformly bounded, we can take a constant $M > 0$ such that

$$\forall x \in \mathcal{X}, \forall t \in [0,1]: \quad \left|x(t)\right| \leq M. \tag{3}$$

Pick a finite set $Y \subset \mathbb{R}^N$ for which

$$\forall z \in B^{\star}(0,3M), \exists y \in Y: \quad |y - z| \leq \epsilon. \tag{4}$$

Now let

$$0 < \alpha \leq 2M \tag{5}$$

be such that $\mu_{\mathrm{uec}}(\mathcal{X}) < \alpha$, that is, there exists $\delta > 0$ such that

$$\forall x \in \mathcal{X}, \forall s,t \in [0,1]: \quad |s-t| < \delta \quad \Rightarrow \quad \left|x(s) - x(t)\right| \leq \alpha. \tag{6}$$

Then choose points

$$0 = t_0 < t_1 < \cdots < t_{2n} < t_{2n+1} = 1,$$

put

$$\begin{aligned}
&I_0 = [0, t_2[,\\
&I_k =]t_{2k-1}, t_{2k+2}[\quad \text{if } k \in \{1, \ldots, n-1\},\\
&I_n =]t_{2n-1}, 1],
\end{aligned}$$

and assume that we have made this choice such that

$$\forall k \in \{0, \ldots, n\}: \quad \operatorname{diam}(I_k) < \delta. \tag{7}$$

Furthermore, for each $(y_0, \ldots, y_{2n+1}) \in Y^{2n+2}$, let $L_{(y_0,\ldots,y_{2n+1})}$ be the $\mathbb{R}^N$-valued map on $[0,1]$ defined by

$$L_{(y_0,\ldots,y_{2n+1})}(t) = \begin{cases} \frac{t_1 - t}{t_1 - t_0} y_0 + \frac{t - t_0}{t_1 - t_0} y_1 & \text{if } t \in [t_0, t_1],\\ \ldots, & \\ \frac{t_{k+1} - t}{t_{k+1} - t_k} y_k + \frac{t - t_k}{t_{k+1} - t_k} y_{k+1} & \text{if } t \in [t_k, t_{k+1}],\\ \ldots, & \\ \frac{t_{2n+1} - t}{t_{2n+1} - t_{2n}} y_{2n} + \frac{t - t_{2n}}{t_{2n+1} - t_{2n}} y_{2n+1} & \text{if } t \in [t_{2n}, t_{2n+1}], \end{cases}$$

and put

$$\mathcal{F} = \left\{L_{(y_0,\ldots,y_{2n+1})} \mid (y_0, \ldots, y_{2n+1}) \in Y^{2n+2}\right\}.$$

Then $\mathcal{F}$ is a finite subset of $\mathcal{C}$. Now fix $x \in \mathcal{X}$ and let $c_{x,k}$ stand for the Chebyshev center of $x(I_k)$ for each $k \in \{0,\dots,n\}$. It follows from (6) and (7) that $\operatorname{diam} f(I_k) \leq \alpha$, and thus, by Theorem 3.1,

$$\forall k \in \{0,\dots,n\}: \quad \sup_{t\in I_k}\bigl|c_{x,k} - x(t)\bigr| \leq \left(\frac{N}{2N+2}\right)^{1/2}\alpha. \tag{8}$$

Let $\tilde{x}$ be the $\mathbb{R}^N$-valued map on $[0,1]$ defined by

$$\tilde{x}(t) = \begin{cases} c_{x,0} & \text{if } t \in [t_0, t_1], \\ \frac{t_2-t}{t_2-t_1}c_{x,0} + \frac{t-t_1}{t_2-t_1}c_{x,1} & \text{if } t \in [t_1, t_2], \\ c_{x,1} & \text{if } t \in [t_2, t_3[, \\ \frac{t_4-t}{t_4-t_3}c_{x,1} + \frac{t-t_3}{t_4-t_3}c_{x,2} & \text{if } t \in [t_3, t_4], \\ \dots, & \\ \frac{t_{2k}-t}{t_{2k}-t_{2k-1}}c_{x,k-1} + \frac{t-t_{2k-1}}{t_{2k}-t_{2k-1}}c_{x,k} & \text{if } t \in [t_{2k-1}, t_{2k}], \\ c_{x,k} & \text{if } t \in [t_{2k}, t_{2k+1}], \\ \frac{t_{2k+2}-t}{t_{2k+2}-t_{2k+1}}c_{x,k} + \frac{t-t_{2k+1}}{t_{2k+2}-t_{2k+1}}c_{x,k+1} & \text{if } t \in [t_{2k+1}, t_{2k+2}], \\ \dots, & \\ \frac{t_{2n-2}-t}{t_{2n-2}-t_{2n-3}}c_{x,n-2} + \frac{t-t_{2n-3}}{t_{2n-2}-t_{2n-3}}c_{x,n-1} & \text{if } t \in [t_{2n-3}, t_{2n-2}], \\ c_{x,n-1} & \text{if } t \in [t_{2n-2}, t_{2n-1}], \\ \frac{t_{2n}-t}{t_{2n}-t_{2n-1}}c_{x,n-1} + \frac{t-t_{2n-1}}{t_{2n}-t_{2n-1}}c_{x,n} & \text{if } t \in [t_{2n-1}, t_{2n}], \\ c_{x,n} & \text{if } t \in [t_{2n}, t_{2n+1}]. \end{cases}$$

Then (8) and Lemma 3.2 yield that

$$\|\tilde{x} - x\|_\infty \leq \left(\frac{N}{2N+2}\right)^{1/2}\alpha. \tag{9}$$

Also, it easily follows from (3), (5), and (9) that $\|\tilde{x}\|_\infty \leq 3M$, and thus (4) allows us to choose $(y_0,\dots,y_{2n+1}) \in Y^{2n+2}$ such that

$$\forall k \in \{0,\dots,2n+1\}: \quad \bigl|y_k - \tilde{x}(t_k)\bigr| \leq \epsilon. \tag{10}$$

Combining (10) and Lemma 3.3 reveals that

$$\|L_{(y_0,\dots,y_{2n+1})} - \tilde{x}\|_\infty \leq \epsilon. \tag{11}$$

Thus, we have found $L_{(y_0,\dots,y_{2n+1})}$ in $\mathcal{F}$ for which, by (9) and (11),

$$\|L_{(y_0,\dots,y_{2n+1})} - x\|_\infty \leq \left(\frac{N}{2N+2}\right)^{1/2}\alpha + \epsilon,$$

which, by the arbitrariness of ϵ, entails that $\mu_{\mathrm{H},\infty}(\mathcal{F}) \leq (\frac{N}{2N+2})^{1/2}\alpha$, and thus, by the arbitrariness of α, the inequality

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2}\mu_{\mathrm{uec}}(\mathcal{X})$$

is established. □

4 Proof of Theorem 1.6

We transport the measure of nonuniform tightness from $\mathcal{P}(\mathcal{C})$ to the collection of c.s.p.s via their probability distributions. Thus, for a set Ξ of c.s.p.s,

$$\mu_{\mathrm{ut}}(\Xi) = \sup_{\epsilon>0} \inf_{\mathcal{F}} \sup_{\xi\in\Xi} \mathbb{P}\left(\xi \notin \bigcup_{y\in\mathcal{F}} B_\infty(y,\epsilon)\right),$$

where the infimum is taken over all finite sets $\mathcal{F} \subset \mathcal{C}$, and

$$B_\infty(y,\epsilon) = \bigl\{x \in \mathcal{C} \mid \|y - x\|_\infty < \epsilon\bigr\}.$$

Before giving the proof of Theorem 1.6, we state three lemmas, which are easily seen to follow from the definitions.

Lemma 4.1 *Let Ξ be a collection of c.s.p.s, and $\alpha \in \mathbb{R}_0^+$. Then the following assertions are equivalent.*

(1) $\mu_{\mathrm{ut}}(\Xi) < \alpha$.
(2) *For each $\epsilon > 0$, there exists a uniformly bounded set $\mathcal{X} \subset \mathcal{C}$ such that*
  (a) $\mu_{\mathrm{H},\infty}(\mathcal{X}) < \epsilon$,
  (b) $\forall \xi \in \Xi$: $\mathbb{P}(\xi \notin \mathcal{X}) < \alpha$.

Lemma 4.2 *Let Ξ be a collection of c.s.p.s, and $\alpha \in \mathbb{R}_0^+$. Then the following assertions are equivalent.*

(1) $\mu_{\mathrm{sub}}(\Xi) < \alpha$.
(2) *There exists a uniformly bounded set $\mathcal{X} \subset \mathcal{C}$ such that*

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{X}) < \alpha.$$

Lemma 4.3 *Let Ξ be a collection of c.s.p.s, and $\alpha \in \mathbb{R}_0^+$. Then the following assertions are equivalent.*

(1) $\mu_{\mathrm{suec}}(\Xi) < \alpha$.
(2) *For each $\epsilon > 0$, there exists a set $\mathcal{X} \subset \mathcal{C}$ such that*
  (a) $\mu_{\mathrm{uec}}(\mathcal{X}) < \epsilon$,
  (b) $\forall \xi \in \Xi$: $\mathbb{P}(\xi \notin \mathcal{X}) < \alpha$.

Proof of Theorem 1.6 Let Ξ be a collection of c.s.p.s. By Theorem 1.4,

$$\sup_{\lambda\in\mathbb{R}_0^+} \mu_{\mathrm{H},\lambda}(\Xi) = \mu_{\mathrm{ut}}(\Xi),$$

whence it suffices to show that

$$\max\bigl\{\mu_{\mathrm{sub}}(\Xi), \mu_{\mathrm{suec}}(\Xi)\bigr\} \le \mu_{\mathrm{ut}}(\Xi) \le \mu_{\mathrm{sub}}(\Xi) + \mu_{\mathrm{suec}}(\Xi).$$

We first establish that

$$\mu_{\mathrm{ut}}(\Xi) \le \mu_{\mathrm{sub}}(\Xi) + \mu_{\mathrm{suec}}(\Xi).$$

Fix $\epsilon > 0$ and $\alpha, \beta \in \mathbb{R}_0^+$ such that

$$\mu_{\mathrm{sub}}(\Xi) < \alpha$$

and

$$\mu_{\mathrm{suec}}(\Xi) < \beta.$$

By Lemma 4.2 there exists a uniformly bounded set $\mathcal{Y} \subset \mathcal{C}$ such that

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{Y}) < \alpha,$$

and by Lemma 4.3 there exists a set $\mathcal{Z} \subset \mathcal{C}$ such that

$$\mu_{\mathrm{uec}}(\mathcal{Z}) < \left(\frac{N}{2N+2}\right)^{-1/2} \epsilon \tag{12}$$

and

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{Z}) < \beta.$$

Put

$$\mathcal{X} = \mathcal{Y} \cap \mathcal{Z}.$$

Then $\mathcal{X}$ is uniformly bounded. Also, by Theorem 1.5 and (12),

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{X}) \leq \left(\frac{N}{2N+2}\right)^{1/2} \mu_{\mathrm{uec}}(\mathcal{Z}) < \epsilon,$$

and, for $\xi \in \Xi$,

$$\mathbb{P}(\xi \notin \mathcal{X}) \leq \mathbb{P}(\xi \notin \mathcal{Y}) + \mathbb{P}(\xi \notin \mathcal{Z}) < \alpha + \beta.$$

We conclude from Lemma 4.1 that

$$\mu_{\mathrm{ut}}(\Xi) < \alpha + \beta,$$

from which the desired inequality follows.

Next, we prove that

$$\max\{\mu_{\mathrm{sub}}(\Xi), \mu_{\mathrm{suec}}(\Xi)\} \leq \mu_{\mathrm{ut}}(\Xi).$$

Fix $\epsilon > 0$ and $\alpha \in \mathbb{R}_0^+$ such that

$$\mu_{\mathrm{ut}}(\Xi) < \alpha.$$

By Lemma 4.1 there exists a uniformly bounded set $\mathcal{X} \subset \mathcal{C}$ such that

$$\mu_{\mathrm{H},\infty}(\mathcal{X}) < \epsilon/2 \tag{13}$$

and

$$\forall \xi \in \Xi: \quad \mathbb{P}(\xi \notin \mathcal{X}) < \alpha.$$

We conclude from Lemma 4.2 that

$$\mu_{\mathrm{sub}}(\Xi) < \alpha.$$

Moreover, by Theorem 1.5 and (13),

$$\mu_{\mathrm{uec}}(\mathcal{X}) \leq 2\mu_{\mathrm{H},\infty}(\mathcal{X}) < \epsilon,$$

and Lemma 4.3 allows us to infer that

$$\mu_{\mathrm{suec}}(\Xi) < \alpha,$$

which finishes the proof of the desired inequality. □

5 Conclusions

In this work, we have quantified the Prokhorov theorem by establishing an explicit formula for the Hausdorff measure of noncompactness (HMNC) for the parameterized Prokhorov metric on the set of Borel probability measures on a Polish space (Theorem 1.4). Furthermore, we have quantified the Arzelà-Ascoli theorem by obtaining upper and lower estimates for the HMNC for the uniform norm on the space of continuous maps of a compact interval into Euclidean N-space, using the Jung theorem on the Chebyshev radius (Theorem 1.5). Finally, we have combined the obtained results to quantify the stochastic Arzelà-Ascoli theorem by providing upper and lower estimates for the HMNC for the parameterized Prokhorov metric on the set of multivariate continuous stochastic processes (Theorem 1.6). This work fits nicely in the research initiated in [6], the aim of which is to systematically study quantitative measures, such as the HMNC, in the realm of probability theory.

Competing interests

The author declares that there are no competing interests.

Author's information

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